

SUBTLETIES IN THE LIGHT-CONE REPRESENTATION

GARY MCCARTOR

Department of Physics, SMU, Dallas, Texas, USA

To produce an isomorphism between the light-cone and equal-time representations some additional formalism beyond that originally proposed for the light-cone representation may sometimes be required. The additional formalism usually involves zero modes and is most likely to affect delicate, high energy aspects of the solution such as condensates. In this talk I will review some of the information which has been obtained in the past few years on these issues with particular emphasis on the Schwinger model as an example.

1 The Light-Cone Representation

To form the light-cone representation for a covariant field theory one proceeds as follows: One specifies initial conditions (the canonical commutation relations) on a characteristic surface, typically $x^+ \equiv x^0 + x^1 = 0$; Negative frequency Fourier modes taken along this surface are taken to be creation operators while positive frequency modes are destruction operators; The bare vacuum is the state destroyed by all the positive frequency modes. The dynamical operators are calculated by integrating densities over the initial value surface, for instance:

$$P^- = \frac{1}{2} \int :T^{-+}: dx^- d^2 x^\perp$$

The relation of this representation to the equal-time representation (initial conditions on the surface $x^0 = 0$) is rather simple for free theories but extremely complicated for interacting theories. Particularly in view of what I shall say below, it is appropriate to provide some reasons for being interested in the light-cone representation. Some commonly given reasons are:

- Boosts are simple (theorem)
- The vacuum is simple (“theorem”)
- Closer to partons (hope)

The first bullet is a theorem; boosts are translations within the initial value surface and thus kinematical and thus simple. The second bullet, the physical vacuum is the bare vacuum, is sometimes given as a theorem but it is not; in particular, as we shall see below, it isn't true. The third bullet lists the expectation that the light-cone bare states are closer to the partons that are observed to make up hadrons than the equal-time bare states. While arguments, some of them quite sensible, are given to defend this expectation, at the moment it is nothing more than a hope.

At the same time that there have been these hopes for advantages to the light-cone representation there have been a number of issues, perhaps one should say puzzles, which have cast doubt as to whether the light-cone representation space could serve the same physics as the equal-time representation space. Some of these puzzles are:

- Degenerate vacua ?
- Condensates ?
- Causality ?

The first bullet is to be seen in conjunction with the second bullet of the earlier list. If there are more than one possible ground states, and sometimes there are, the bare vacuum could be one of the possibilities but not all. A somewhat less concrete puzzle is presented by condensates: in the equal-time representation it is usual to think of them as vacuum phenomena. If they do not appear in that way in the light-cone representation, how do they appear? Finally, What about the general question of causality — since the points on our initial value surface are causally connected, can we really specify initial conditions on them without great care to avoid contradictions?

2 Zero Modes

Perhaps the greatest interest in the light-cone representation stems from the idea that the physical vacuum is the bare vacuum. At the same time this idea leads to the most direct contradictions as discussed in the previous section. Let us therefore review the argument that $|\Omega\rangle = |0\rangle$. The argument begins with the statement that the operator P^+ in the interacting theory is the same as in free theory. The physical vacuum should satisfy $P^+|\Omega\rangle = 0$ and, at least for theories which can be initialized on $x^+ = 0$, there is only one such state — the bare vacuum. That's the essence of the argument. The argument that $P^+ = P_{FREE}^+$ may be given two different ways. The first way just calculates it by integrating a density over the initial value surface:

$$P^+ = \frac{1}{2} \int T^{++} dx^-$$

$$T^{++} = \sum_{\phi} \frac{\partial \phi}{\partial x_+} \frac{\partial \mathcal{L}}{\partial (\partial_+ \phi)} - g^{++} \mathcal{L}$$

Since $g^{++} = 0$ we get $P^+ = P_{FREE}^+$. More pertinent to the remarks below is the following argument: The fields must satisfy the Heisenberg relation

$$\partial_- \phi = \frac{i}{2} [P^+, \phi]$$

This relation may be checked using values entirely within the initial value surface. But since we initialize the fields to be isomorphic to free fields on that surface, we know that there

$$\partial_- \phi = \frac{i}{2} [P_{FREE}^+, \phi]$$

Thus if the fields form an irreducible set, which basically means we have a well posed physics problem, the only modification to P_{FREE}^+ we can make would be to add a multiple of the identity. But, of course, that is exactly the point: for many cases one cannot completely specify the physics problem by providing information on the characteristic $x^+ = 0$ ¹. If we consider the free, massless Fermi field in two dimensions with periodicity conditions to induce a discrete set of modes we find

$$\begin{aligned}\psi_+(x^+, x^-) &= \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} b(n) e^{-ik_-(n)x^-} + d^*(n) e^{ik_-(n)x^-} \\ \psi_-(x^+, x^-) &= \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} \beta(n) e^{-ik_+(n)x^+} + \delta^*(n) e^{ik_+(n)x^+}\end{aligned}$$

Here we see that the entire field, ψ_- , is composed of zero modes — functions only of x^+ . The ψ_- field cannot be initialized on the surface $x^+ = 0$. Note also that the formula for P^- given above will not suffice; we need

$$P^- = \frac{1}{2} \int :T^{-+}: dx^- + \frac{1}{2} \int :T^{--}: dx^+$$

The β and δ modes are examples of unconstrained zero modes — that is, they are true degrees of freedom of the system. It is worth remarking that there is another type of zero mode called a constrained zero mode. These are not true degrees of freedom but are related to the degrees of freedom by constraint relations. An example is furnished by ϕ^4 theory. In two dimensions the equation of motion is

$$(4\partial_+ \partial_- + \mu_0^2) \phi = -\lambda \phi^3$$

Even though the ϕ field does not have a zero mode in free theory, the right hand side of the above equation does have one; thus in the interacting theory the field ϕ does have a zero mode which goes to zero when λ goes to zero and which is determined in terms of the degrees of freedom by the equation of motion. This type of zero mode was first discussed by Maskawa and Yamawaki². Such

a zero mode is responsible for the condensate in wrong-sign ϕ^4 theory³⁴. Thus we see that in fact aspects of the solution which appear as vacuum structure in the equal-time representation can appear in other ways in the light-cone representation — sometimes as the development of a constrained zero mode. The constrained zero modes cannot lead to the existence of degenerate ground states, however. For that we shall require the unconstrained zero modes such as those of the ψ_- field.

3 The Schwinger Model

Let us now illustrate some of these ideas with the best studied case: the Schwinger model⁵. First consider the problem of calculating the operator P^+ . We shall choose antiperiodic boundary conditions for the Fermi fields and initialize ψ_+ on $x^+ = 0$ and ψ_- on $x^- = 0$

$$\begin{aligned}\psi_+(0, x^-) &= \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} b(n) e^{-ik_-(n)x^-} + d^*(n) e^{ik_-(n)x^-} \\ \psi_-(x^+, 0) &= \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} \beta(n) e^{-ik_+(n)x^+} + \delta^*(n) e^{ik_+(n)x^+}\end{aligned}$$

The general considerations based on the Heisenberg equations suggest that modes from the ψ_- field might mix with P^+ . An explicit calculation shows that this expectation is realized. We find

$$P^+ = \frac{1}{2} \int_{-L}^L :2i(\psi_+^* \partial_- \psi_+ - \partial_- \psi_+^* \psi_+) : dx^-$$

This equation seems to have the same form as free theory but the symbols mean something different. To maintain gauge invariance we must define

$$T^{++} = 2i \lim_{\epsilon^- \rightarrow 0} \left(e^{-ie \int_x^{x+\epsilon^-} A_-^{(-)} dx^-} \psi_+^*(x+\epsilon^-) \partial_- \psi_+(x) e^{-ie \int_x^{x+\epsilon^-} A_-^{(+)} dx^-} - \text{C.C.} - \text{V.E.V.} \right)$$

With this definition a simple calculation gives

$$P^+ = P_{FREE}^+ - \left(\frac{1}{2} A^+\right)^2$$

The operator A^+ is an example of a constrained zero mode. To understand how to calculate it properly we shall require some discussion. If we take the zero mode component of the Maxwell equation

$$\frac{\partial^2 A^-}{\partial x^{-2}} = -\frac{1}{2} J^+$$

we get

$$\frac{1}{2L}Q_+ = \frac{e^2}{2\pi}A^+ \quad (WRONG)$$

If this relation were correct we would have an immediate problem in that Q_+ is composed of modes from the ψ_+ field which are the ones we saw above could NOT be mixed with P^+ without producing an immediate conflict with the Heisenberg equations. To understand why the relation is wrong we can consider a problem in classical E&M ⁶. A charge density which is constant everywhere in space cannot create a field. Among other things there would be no preferred direction for the field to point. On the other hand, if $\rho(0)$ is a charge density constant in space, the Maxwell equation

$$\frac{\partial E}{\partial x} = \rho(0)$$

does not have zero as a possible solution. The field really couples to the current J' where

$$J'^0 = J^0 - J^0(0) \quad ; \quad J'^1 = J^1$$

With this correction the solution for A^+ becomes

$$\frac{1}{2L}Q_- = \frac{e^2}{2\pi}A^+$$

Which then gives

$$P^+ = \frac{1}{2} \int_{-L}^L :2i(\psi_+^* \partial_- \psi_+ - \partial_- \psi_+^* \psi_+) : dx^- = P_{FREE}^+ - \frac{1}{4Lm^2} Q_-^2$$

Now the operators mixed with P^+ come from the ψ_- field and there is no conflict with the Heisenberg equations. The correction to P^+ — necessary to maintain gauge invariance — causes some states which are split in free theory to become degenerate ground states in the interacting theory; linear combinations of them provide the expected θ states familiar in the Schwinger model. The physical subspace of the Schwinger model is chargeless and so defined by

$$\frac{1}{2}(Q_+ + Q_-)|p\rangle = 0$$

So we may write

$$Q_- \approx Q_+$$

From this point of view the correction from the equation labeled *WRONG* to the correct result can be viewed as substituting an operator for its weak

equivalent. In the case of the Schwinger model we completely understand the physical basis for that requirement; in some other cases we find that we must make such a substitution, again to avoid conflict with the Heisenberg equations, and we do not as completely understand the physical basis. General rules on the subject are not known and we currently have to treat each case separately.

A full operator solution can be given with these boundary conditions. Due to space limitations I shall not do so here but will list some of the things one must do to formulate such a solution⁵

- KEEP $\psi_-(x^+ + 2L) = -\psi_-(x^+)$
- COUPLE TO J'
- KEEP A^+ — A CONSTRAINED ZERO MODE
- MAINTAIN GAUGE INVARIANCE
- PHYSICAL SUBSPACE $D(n)|P\rangle = 0$
- CALCULATE P^- USING x^+ and x^-

The next to last bullet refers to the fact that a more stringent physical subspace condition must be imposed in this gauge than just the chargeless one. Some properties of the solution one then finds include⁵

- SPECTRUM
- ANOMALY
- θ -VACUUM STRUCTURE
- MUCH SIMPLER SOLUTION ($a^*(n) = C^*(n)$)
- $\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{1}{L} \cos \theta$

The first three bullets are in accord with standard calculations. The fourth bullet makes definite the idea that the light-cone bare states are closer to partons: the light-cone fusion operators create the Schwinger particles. The last bullet shows an unexpected result: the condensate goes to zero as L goes to infinity. That aspect is not due either to the choice of light-cone gauge or to the choice of quantization surface but rather to the choice of boundary conditions: the continuum the light-cone gauge solution has the expected value of the condensate whatever quantization surface is used. The imposition of the periodicity conditions has too abruptly removed the small p^+ region of the spectrum and the very singular behavior there, which is responsible for the condensate, is not recovered as L becomes large.

I believe that the behavior of the condensate we have just seen may be related to a puzzle in the 't Hooft⁷ model (Large N QCD_{1+1}). 't Hooft solved the model and gave the (dressed) propagator. By Fourier transforming the propagator to coordinate space one may easily determine that it implies no condensate. On the other hand Zhitnitsky⁸ has shown that the spectrum and

wavefunctions 't Hooft derived from his propagator imply that

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = -N \sqrt{\frac{g^2 N}{12\pi}}$$

't Hooft used light-cone gauge and light-cone quantization to formulate his solution. While he did not use periodicity conditions, his method of regulating the small p^+ region was pretty abrupt and had much the same effect. The situation is further complicated by the fact that T.T. Wu⁹ has given a different propagator, the difference being solely due to the treatment of the small p^+ region. Some recent discussion of those issues is in Bassetto and Griguolo¹⁰.

4 Discussion

In view of the difficulties I have presented one might well ask: “can all this really be worth it?” I do not yet know; but

- Boosts are simple

In cases where the answer is known

- The vacuum is simpler

Not completely trivial but much simpler. And at least for the Schwinger model it is true that the light-cone bare states are

- Closer to partons

References

1. G. McCartor, *Z. Phys. C* **41**, 271 (1988).
2. T. Maskawa and K Yamawaki, *Prog. Theor. Phys.* **56**, 270 (1976).
3. D. G. Robertson, *Phys. Rev. D* **47**, 2549 (1993).
4. T. Heinzl, S. Krusche and E. Werner, *Phys. Lett. B* **256**, 55 (1991).
5. G. McCartor, *Z. Phys. C* **64**, 349 (1994).
6. G. McCartor, *Z. Phys. C* **52**, 611 (1991).
7. G. 't Hooft, *Nucl. Phys. B* **75**, 461 (1974).
8. A. Zhitnitsky, *Phys. Lett. B* **165**, 405 (1985).
9. T.T. Wu, *Phys. Lett. B* **71**, 142 (1977).
10. A. Bassetto and L. Griguolo, **hep-th/9603084**.